Lecture 12

In this lecture we complete our study of matrices.

Let $n \geq 1$, n an integer.

The matrix

$$I_n = \begin{pmatrix} 1 & & \bigcirc \\ & 1 & \\ & & \ddots & \\ \bigcirc & & & 1 \end{pmatrix}$$

is the identity matrix of order n. It has 1 everywhere in the leading diagonal (top left to bottom right), and 0 everywhere else.

$$O_n = \begin{pmatrix} 0 & & \bigcirc \\ & 0 & \\ & & \ddots & \\ \bigcirc & & & 0 \end{pmatrix}$$

is the zero matrix of order n.

Sometimes we just write I or O instead of I_n or O_n.

$$O + A = A = A + O$$

whenever O & A have the same order.

$$O \cdot A = O \left(= A \cdot O \text{ possibly} \right)$$

whenever the product exists.

$IA = A$ whenever the product exists

$AI = A$ " " " "

"Additive inverse" or <u>Negative</u>.

Given A, if we negate all the entries we get $-A$.

Then $A + (-A) = O$
$= -A + A$.

<u>(Multiplicative) Inverse</u>

Given a $\underbrace{\text{square}}_{\text{same no. of rows as columns}}$ matrix A:

Sometimes there exists another square matrix

$$A^{-1}$$

with the same order as A and such that

$$AA^{-1} = I$$
$$= A^{-1}A.$$

Then A^{-1} is called the <u>inverse</u> of A, and A is said to be <u>invertible</u>.

How do we know when A^{-1} exists?

It's easy in the (2×2)-case. Even the (3×3)-case is much harder.

Determinants

Associated with any square matrix A is a special number called the <u>determinant</u> of A.

It has a role in providing info about the extent to which A operates on a space of vectors to cause an expansion or contraction of that space. (We ignore the details.)

It is denoted by

$$|A| \quad \underline{\text{or}} \quad \det A$$

$$\underline{\text{or}} \quad \det(A).$$

In the two-by-two case, a general matrix is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the determinant is $|A|$, and we write

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}.$$

Theorem

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then $\boxed{A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

$$\boxed{= \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

provided that $ad-bc \neq 0$.

Partial proof

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

So $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ satisfies the property needed for it to be an inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It can be shown that there's only one inverse. So this is it.

For matrices of higher order we still divide by the determinant but the formula for the inverse is much more complicated.

A matrix is called <u>singular</u> if its determinant is zero.

So we've seen that <u>a square matrix is invertible if and only if it is nonsingular</u>.

Examples

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - (-3) \cdot 2$$

$$= 4 - (-6)$$

$$= 4 + 6 = 10$$

$$\therefore \; A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$$

Check: $\frac{1}{10} \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$

$$= \frac{1}{10} \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$